- **Riemann Sums:**
  - Evaluates integrals by treating them as rectangles of subintervals.
  - Approximating using Riemann sums:
    - $\int f(x)dx = \sum_{i=1}^{n} f(x_i^*)\Delta x$ n = number of subintervals.
    - Divide the region up into rectangles and find their areas, then add.
    - Lower Riemann sum: uses the lower boundary of a subinterval for x_i^*
    - Upper Riemann sum: uses the upper boundary of a subinterval for x_i^*
    - Midpoint Riemann sum: uses the midpoint of a subinterval for x_i^*
  - Exactly evaluating an integral using a Riemann Sum:
    - Divide into infinitely many subintervals: $\int f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$
    - By extension: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)\left(\frac{b-a}{n}\right)$
    - A limit of a Riemann sum is the most fundamental definition of an integral.
    - It may be necessary to flip the order of integration (i.e. rewrite with respect to y)
  - Summation formulas (derived from discrete mathematics):

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

- **Trapezoidal Rule**
  - Approximates integrals by dividing the region into trapezoids (i.e. first-degree linear approximation)
  - Know the area of a trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$
    - For our purposes: $A = \frac{1}{2}(f(x_1) + f(x_2))(x_2 - x_1)$
  - Let the interval [a, b] be divided into n subintervals, with the i^{th} subinterval with lower and upper bounds being x_{i-1} and x_i respectively. By the Trapezoidal Rule:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{1}{2}(f(x_{i-1}) + f(x_i))(x_{i-1} - x_i)$$

- **Simpson's Rule**
  - Approximates integrals using a second-degree Taylor polynomial to approximate the curve on each interval (this can be extended to n degrees for better accuracy)
  - Let the interval [a, b] be divided into n subintervals, with the i^{th} subinterval with lower and upper bounds being x_{i-1} and x_i respectively.

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{1}{6}\left(f(x_{i-1}) + 4f\left(\frac{x_{i-1} + x_i}{2}\right) + f(x_i)\right)(x_{i-1} - x_i)$$